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A series study of a mixed-spin $S = (\frac{1}{2}, 1)$ ferrimagnetic Ising model

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Abstract
We use both high- and low-temperature series expansions to investigate the phase diagram of a ferrimagnetic mixed-spin $S = (1/2, 1)$ Ising model on the square lattice, including an on-site anisotropy term on the $S = 1$ sites. Evidence is found for a first-order transition for large negative anisotropy, and hence for the existence of a tricritical point. The model, with nearest neighbour interactions only, does not appear to have a ferrimagnetic compensation point.

1. Introduction

Mixed-spin Ising models have been studied for some time as simple models of ferrimagnetism, and there has been renewed interest recently in connection with compensation points. These are temperatures, below the critical point, where the sublattice magnetizations exactly cancel, with zero total moment. As the temperature is tuned through such a point the total moment changes sign. In this context Ising models have the virtue of being exactly solvable in special cases [1] or solvable to high numerical precision by Monte Carlo [2, 3] or other methods.

The present paper uses high- and low-temperature series expansions to study a particular mixed-spin model. A bipartite square lattice has $S = \frac{1}{2}$ spins on one sublattice (denoted A) and $S = 1$ spins on the other (denoted B), with nearest neighbour interactions and a single-ion anisotropy term on the $S = 1$ sites. The Hamiltonian of our model is

$$H = -J \sum_{\langle ij \rangle} \sigma_i S_j - h_A \sum_{i \in A} \sigma_i - h_B \sum_{j \in B} S_j - D \sum_{j \in B} S_j^2 \qquad (1)$$

where $\sigma_i = \pm 1$, $S_j = 1, 0, -1$, h_A and h_B are fields on the two species and D is the anisotropy. Note that we choose $\sigma_i = \pm 1$ rather than $\pm\frac{1}{2}$ as done by some other authors, This simply amounts to a rescaling of the exchange constant J. Note also that we have written H in the form of a ferromagnet ($J > 0$) but this is equivalent to a ferrimagnet by a simple spin reversal on either sublattice.

The model, and a schematic phase diagram, are shown in figure 1. For $D/J > -4$ there will be a transition line separating the low-temperature ferromagnetic phase from the

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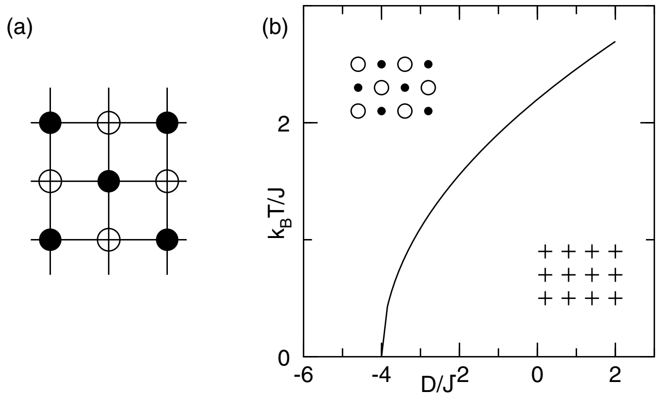

Figure 1. The ferrimagnetic model: (a) lattice showing the A sites as ● and the B sites as ○. (b) Schematic of phase diagram with illustration of ground states. For $D/J < -4$, the B spins are in the 'zero' state (○) and the A spins are completely decoupled (•), and thus disordered even at $T = 0$.

high-temperature disordered phase. At $D/J = -4$ the ferromagnetic and $S = 0$ states will have equal energy and for all $D/J < -4$ the ground state is infinitely degenerate. A mean-field treatment [4] predicts a tricritical point at $D/J = -3.720$, and thus a first-order transition in the range $-4 < D/J < -3.720$. The mean-field solution also has a compensation point for cases with $-4 < D/J \leqslant -2\ln 6$ $(-3.3535\ldots)$. However, neither of these features was observed in a recent study [2] based on Monte Carlo and numerical transfer matrix calculations.

The mixed-spin model has also been studied previously by high-temperature [5] and low-temperature series [6] but not in the context of ferrimagnetism and compensation points, nor with the inclusion of a single-ion anisotropy term. In the present work we extend these series substantially and search for evidence of tricritical and/or compensation points.

2. Series expansions

In this section we describe the series derivation and present some of the raw data. The procedures are standard and described in both overview reference works [7, 8] and specific articles [9–12] and so we will omit much of the detail.

2.1. High-temperature expansions

Expansions have been derived for the zero-field free energy, f, and susceptibility, χ, in powers of $K = \beta J$ to order K^{16} with $\beta = 1/k_B T$. The expansions can be written in the form

$$-\beta f(H, \beta D) = \frac{1}{N}\ln Z = \frac{1}{2}\ln\left(\frac{2}{1-p}\right) + \sum_{r=2}^{\infty} A_r(p) K^r \tag{2}$$

and

$$\chi = \sum_{r=0}^{\infty} G_g(p) K^r \tag{3}$$

where

$$p = 2\mathrm{e}^{\beta D}/(1 + 2\mathrm{e}^{\beta D}) \tag{4}$$

and $A_r(p)$ and $G_r(p)$ are polynomials in p. These polynomials are given in appendix A. We note that earlier series work [5] gives the coefficients A_r to order 10 and G_r to order 7 only for the case $p = \frac{2}{3}$ $(D = 0)$. Our extended results confirm these.

A few technical comments are in order. The free energy series was derived by the direct method involving both connected and disconnected graphs with single and double bonds. For $p = 1$ $(D = \infty)$ the $S = 0$ states are suppressed and the series should reduce to the known spin-$\frac{1}{2}$ Ising series [9]. This provides a partial check on the correctness of our results. The susceptibility series was obtained using the 'vertex-renormalized linked-cluster' method [10]. To obtain the susceptibility with respect to a uniform external field h we write the specific fields as $h_A = m_A h$ and $h_B = m_B h$ where $m_A = \frac{1}{2}$ and $m_B = 1$ are the relative magnetic moments. In this case, the $p = 1$ limit is not exactly the known spin-$\frac{1}{2}$ susceptibility series, because of the different moments, but odd and even coefficients in the expansion of the mixed-spin susceptibility are related to the corresponding coefficients in the spin-$\frac{1}{2}$ case by factors of $2m_A m_B = 1$ and $m_A^2 + m_B^2 = \frac{5}{4}$, respectively. This has provided an additional check on our series. We have taken every care to avoid errors, but even with these checks the possibility of small errors in the high-order coefficients cannot be excluded.

In section 3 of the paper, we will use the susceptibility series to obtain the locus of the line of critical points of the model and use the free energy series to search for evidence of a first-order transition, using 'free energy matching'.

2.2. Low-temperature expansions

Expansions have been derived for the zero-field free energy and sublattice magnetizations, $M_A = \frac{1}{2}\langle\sigma\rangle$ and $M_B = \langle S\rangle$, in powers of the variable $u = \exp(-2J/k_\mathrm{B}T) = \exp(-2K)$. The expansions take the form

$$-\beta f(u, \beta D) = 4\beta J + \beta D + \sum_{r=2}^{\infty} \psi_r(y)\, u^r \tag{5}$$

$$2M_A = 1 - \sum_{r=4}^{\infty} \mu_r^{[A]}(y)\, u^r \tag{6}$$

$$M_B = 1 - \sum_{r=4}^{\infty} \mu_r^{[B]}(y)\, u^r \tag{7}$$

where $y = \exp(-\beta D)$. The quantities $\psi_r(y)$, $\mu_r^{[A]}(y)$ and $\mu_r^{[B]}(y)$ are polynomials in y and are given in appendix B to order $r = 19$. Thus the resulting series can be obtained to order u^{19} for any fixed value of βD.

These series were derived using the method of partial generating functions (PGF) [7, 11] where each PGF corresponds to a fixed number of spin excitations on the A sublattice. Bowers and Yousif [6] have shown that these can be obtained from the corresponding PGFs for the pure spin-$\frac{1}{2}$ problem. Using the published PGFs to F_7 [11] and augmenting these with the lower-order (in u) contributions to F_8, F_9 and F_{10} allows a 'temperature grouping' to order u^{19} to be computed. A check is available for the case $y = 1$ $(D = 0)$ as discussed below. The series will be used in section 3 to explore the region $D/J \leqslant -3$ where the most interesting physics is expected.

An alternative, usually more powerful, method of obtaining low-temperature series expansions, particularly in two dimensions, is the finite lattice method (FLM) [12]. We have adapted this method to the mixed-spin problem and obtained series for Z (rather than for $\ln Z$) and the sublattice magnetizations for the case $D = 0$. The FLM combinatorics for treating

Table 1. Estimates of critical temperature kT_c/J for various D/J from analysis of high-temperature susceptibility series. The values of K_c for χ and M are from high-temperature susceptibility series and low-temperature magnetization series, respectively.

βD	$K_c\ (\chi)$	$K_c\ (M)$	kT_c/J	D/J
0.940	0.4713(1)	—	2.122(1)	1.944
0.488	0.4870(3)	—	2.053(1)	1.002
0	0.5120(5)	0.5199	1.953(2)	0
−0.553	0.5535(10)	—	1.807(3)	−0.999
−0.765	0.5735(10)	0.5725	1.744(3)	−1.334
−1.260	0.630(1)	0.6285	1.587(3)	−2.000
−1.935	0.729(2)	—	1.372(3)	−2.654
−2.450	0.815(5)	—	1.227(6)	−3.006
−5.000	1.36(1)	—	0.735(5)	−3.67

sublattices is a slightly simplified case of that described for the checkerboard lattice [17]. The series are given in appendix C. These data provide a check on the shorter general series for the case $y = 1$.

It is possible to include the single-ion anisotropy in FLM calculations. However, issues of numerical precision and storage requirements mean that this is most easily done for single-variable series defined by $y = u^{n/m}$ so that $D/J = 2n/m$ where $n = 4k - 2m$ and k and m are small integers. We have derived series (of varying length) for a number of choices of n/m. These series are valuable, in conjunction with the high-temperature susceptibility, in locating the critical line. Magnetization series for $D/J = 0, -4/3, -2$ were used to obtain estimates of the critical temperature, given in table 1. We have also calculated series for the cases $D/J = 4, 8, 16$. However, it has not been possible to explore the region $D/J \leqslant -3$ in this way.

3. Results

We now turn to the results obtained by analysis of the various series.

3.1. The critical line

As shown in the schematic phase diagram (figure 1(b)), the model will have a line of critical points in the (T, D) plane. Critical temperatures are usually obtained most precisely from analysis of the high-temperature susceptibility series. We choose fixed values of βD, obtain the critical point K_c from a standard Padé approximant analysis [13] of both logarithmic derivative series and the series for $\chi^{4/7}$ which should have a simple pole, and obtain the corresponding values of D/J as $\beta D/K_c$. In table 1, we give estimates of $k_B T/J$. The quoted uncertainties are, as usual, confidence limits based on consistency and apparent convergence among a range of approximants. The resulting critical curve is shown in figure 2. For $\beta D \lessapprox -3.0$ the susceptibility series becomes more and more irregular and the analysis more problematic. This is due to the presence of interfering singularities and we are unable to continue the line to $D/J = -4$ where $T_c = 0$. If the transition in this region becomes first-order, as we argue below, the high-temperature susceptibility series will not in any case diverge at the true transition temperature but at a spinodal line within the ordered phase. The critical line obtained in the present study agrees well with earlier results from a transfer matrix approach [2] and lies substantially below the mean-field curve [4] which is shown in figure 2 for comparison.

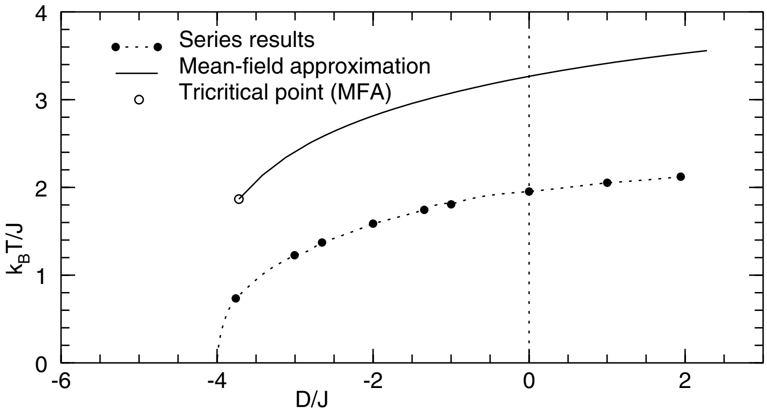

Figure 2. Critical temperature of the mixed-spin model as function of anisotropy, estimated from high-temperature series. The estimated errors are smaller than the plotting symbols. For comparison, the mean-field result (MFA) [4] is also shown, with its predicted tricritical point (TCP).

3.2. First-order transitions

Mean-field theory [4] predicts the existence of a tricritical point at $D/J = -3.720$ and hence a first-order transition in the range $-4 < D/J < -3.72$. While series expansions are not, in general, well-suited to identifying first-order transitions, they have been used successfully in the past in this context [14–16]. The idea is to compare free energies obtained from high- and low-temperature series and see whether the curves join smoothly or with different slopes, the latter being characteristic of a first-order transition.

We have carried out such an analysis for the present model and show, in figure 3, estimated free energy curves for four cases: $D/J = -2, -3, -3.2$ and -3.6. In the first two cases the curves appear to join smoothly and are virtually indistinguishable over ranges of K, $(0.6, 0.67)$ and $(0.7, 0.9)$ that in each case encompass the estimated critical points: $K_c \approx 0.63$ and $K_c \approx 0.8$. It is clearly not possible to estimate K_c with any accuracy in this way. In the other cases, $D/J = -3.2$ and -3.6, there is a clear indication of first-order transitions at $K \approx 0.83$ and $K \approx 1.3$. If this conclusion is valid there will be a tricritical point around $D/J = -3.1$, although it is difficult to locate it to high precision in this way. The $D/J = -3.1$ case (not shown) shows ambiguous behaviour, indicating that much longer series, giving smaller uncertainties in βf, would be needed to locate the tricritical point more precisely. The transfer matrix and Monte Carlo studies [2] did not find a tricritical point in this model.

3.3. Compensation points

As mentioned previously, the mean-field study of this model [4], for the ferrimagnetic case $J < 0$, predicts the existence of a compensation point in the ordered phase where the total moment $m = \frac{1}{2}\langle\sigma\rangle + \langle S\rangle$ vanishes. We have used the low-temperature series of section 2.2 to evaluate both sublattice magnetizations, $M_A = \frac{1}{2}\langle\sigma\rangle$ and $M_B = \langle S\rangle$, as functions of T for various values of the anisotropy D/J. Typical results are shown in figure 4, for $D/J = -2.0, -3.6$. Note that we consider the equivalent ferromagnetic system in which the ferrimagnetic compensation point is signalled by $M_A = M_B$.

The magnetization of the A sublattice remains close to the saturation value until close to T_c where it drops sharply. This behaviour follows that of the simple Ising model. The B sublattice

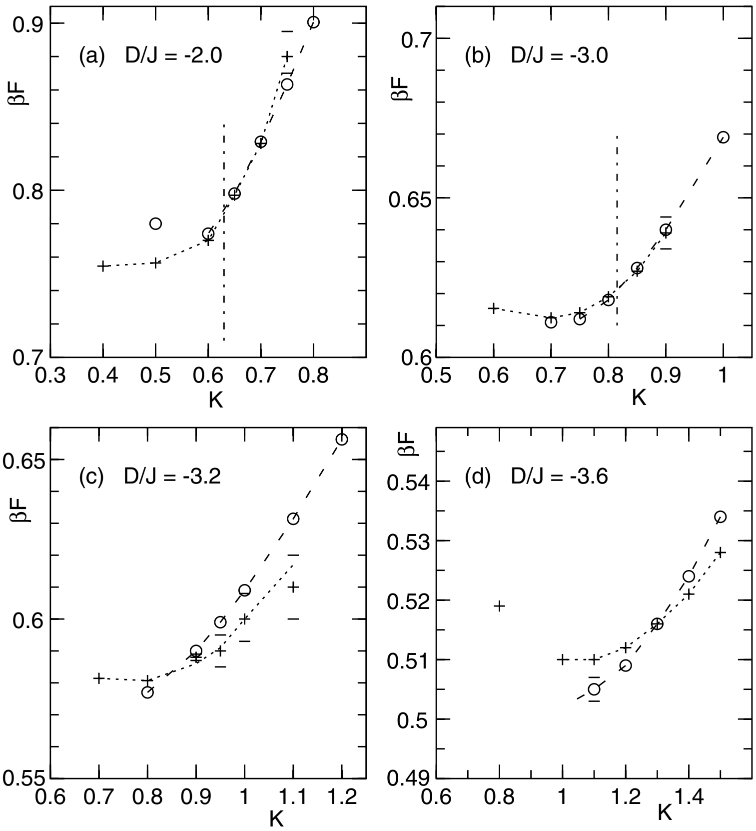

Figure 3. Matching of high-temperature, +, and low-temperature, ○, series estimates of free energies for (a) $D/J = -2.0$, (b) $D/J = -3.0$, (c) $D/J = -3.2$ and (d) $D/J = -3.6$. In cases (a) and (b), the branches meet smoothly, whereas in cases (c) and (d), the transition appears to be first order. Unless shown, the error bars are no larger than the plotting symbols in the figure. Lines are guides for the eye. The vertical chain line in parts (a) and (b) indicates the critical point estimated from high-temperature susceptibility series.

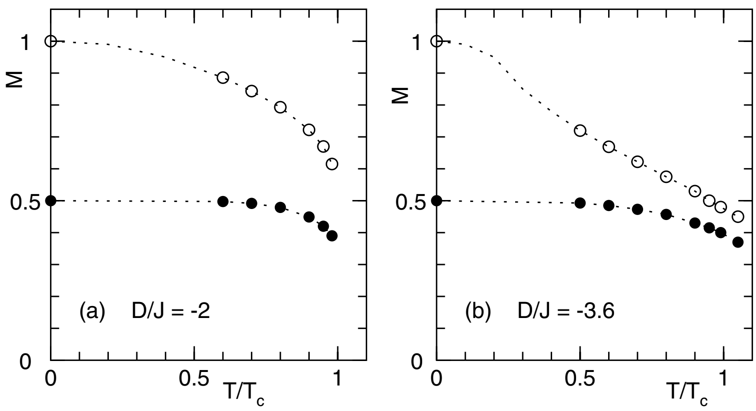

Figure 4. Sublattice magnetizations M_A ● and M_B ○ as functions of temperature for (a) $D/J = -2.0$ and (b) $D/J = -3.6$. No compensation point is seen in either case.

magnetization, on the other hand, has a more complex temperature dependence and develops a point of inflection for D/J approaching -4. Physically this is due to an increasing population of the $S = 0$ state with increasing temperature and is the precursor to a compensation point (where M_B would fall below M_A). However, we find no compensation point, down to at least $D/J = -3.9$.

4. Conclusions

We have used high- and low-temperature series expansions to investigate a mixed-spin $S = (\frac{1}{2}, 1)$ Ising model with nearest neighbour interactions on the square lattice. A single-ion anisotropy term is included in the $S = 1$ sublattice. Our main motivation has been to search for the existence of a tricritical point and/or compensation point, both of which are predicted by mean-field theory [4], but neither of which was found by a more reliable Monte Carlo/transfer matrix study [2]. In the process we have substantially extended the series for this model.

Although we find no compensation point, the magnetization of the $S = 1$ sublattice shows an anomalous temperature dependence for large negative anisotropy D, which if magnified would lead to a compensation point. Rather surprisingly, we do find evidence for a first-order transition at large negative D and hence for a tricritical point that is not seen in previous work [2]. This suggests that further study of the model is warranted.

The previous study [2] found that inclusion of a next-nearest neighbour interaction between the $S = \frac{1}{2}$ spins did yield a compensation point. Such an interaction will stiffen the order on the A sublattice to higher temperatures. We have not included such an interaction in our series work although it could be done with some effort.

Series expansion methods have not, to our knowledge, been used previously in looking for compensation points in ferrimagnetic models. However, as we demonstrate here, they can potentially be a powerful approach. Provided that reasonably long magnetization series can be derived, the resulting analysis can be quite precise, as the compensation point, unlike the critical point, is not a singularity.

Acknowledgments

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Appendix A. High-temperature polynomials

Free energy equation (2)

$A_2(p) = p$

$A_4(p) = (5/6)p$

$A_6(p) = (17/45)p + p^2 - (2/3)p^3$

$A_8(p) = (13/126)p + (7/4)p^2 - (5/3)p^3 + (3/2)p^4$

$A_{10}(p) = (257/14\,175)p + (22/15)p^2 - (29/45)p^3 + 4p^4 - (4/5)p^5$

$A_{12}(p) = (205/93\,555)p + (1453/1890)p^2 + (2153/1134)p^3 + (28/5)p^4 + (8/3)p^5 + p^6$

$$A_{14}(p) = (8194/42\,567\,525)p + (1406/4995)p^2 + (9383/2835)p^3 + (4635/567)p^4 + (254/15)p^5 + 8p^6 + (8/7)p^7$$

$$A_{16}(p) = (3277/255\,405\,150)\,p + (15\,949/207\,900)\,p^2 + (177\,043/62\,370)\,p^3 + (161\,513/12\,600)\,p^4 + (12\,449/315)\,p^5 + (869/15)\,p^6 + (14/3)\,p^7 + (45/4)\,p^8.$$

Susceptibility equation (3)

$$G_0(p) = (1/8) + (1/2)\,p$$
$$G_1(p) = 2p$$
$$G_2(p) = (5/2)\,p + 5p^2$$
$$G_3(p) = (10/3)\,p + 14p^2$$
$$G_4(p) = (17/6)\,p + (86/3)\,p^2 + 26p^3$$
$$G_5(p) = (34/15)\,p + 48p^2 + 70p^3$$
$$G_6(p) = (13/9)\,p + (1075/18)\,p^2 + (1205/6)\,p^3 + 120p^4$$
$$G_7(p) = (52/63)\,p + (1064/15)\,p^2 + (1114/3)\,p^3 + 326p^4$$
$$G_8(p) = (257/630)\,p + (20\,921/315)\,p^2 + (6227/10)\,p^3 + (2303/2)\,p^4 + 540p^5$$
$$G_9(p) = (514/2835)\,p + (1648/27)\,p^2 + (7900/9)\,p^3 + (6928/3)\,p^4 + 1434p^5$$
$$G_{10}(p) = (41/567)\,p + (262\,741/5670)\,p^2 + (412\,487/378)\,p^3 + (28\,385/6)\,p^4 + (12\,133/2)\,p^5 + 2328p^6$$
$$G_{11}(p) = (164/6237)\,p + (18\,232/525)\,p^2 + (167\,818/135)\,p^3 + (112\,784/15)\,p^4 + 12698p^5 + 6164p^6$$
$$G_{12}(p) = (4097/467\,775)\,p + (10\,387\,387/467\,775)\,p^2 + (35\,464\,649/28\,350)\,p^3 + (64\,646\,307/5670)\,p^4 + (91\,642/3)\,p^5 + 30346p^6 + 9724p^7$$
$$G_{13}(p) = (16\,388/6081\,075)\,p + (62\,608/4455)\,p^2 + (96\,868/81)\,p^3 + (74\,990\,194/4995)\,p^4 + (788\,602/15)\,p^5 + (196\,214/3)\,p^6 + 25\,730p^7$$
$$G_{14}(p) = (6554/8513\,505)\,p + (66\,025\,327/8513\,505)\,p^2 + (235\,394/231)\,p^3 + (498\,094/27)\,p^4 + (1953\,221/21)\,p^5 + (361\,211/2)\,p^6 + (288\,405/2)\,p^7 + 40392p^8$$
$$G_{15}(p) = (26\,216/127\,702\,575)\,p + (18\,455\,024/4343\,625)\,p^2 + (130\,902\,236/155\,925)\,p^3 + (98\,021\,494/4725)\,p^4 + (42\,746\,416/315)\,p^5 + (14\,770\,084/45)\,p^6 + (956\,302/3)\,p^7 + 106\,216p^8$$
$$G_{16}(p) = (65\,537/1277\,025\,750)\,p + (1315\,330\,466/638\,512\,875)\,p^2 + (10\,192\,275\,347/16\,372\,125)\,p^3 + (160\,684\,229/7425)\,p^4 + (122\,425\,943/630)\,p^5 + (138\,933\,001/210)\,p^6 + (14\,870\,686/15)\,p^7 + 668\,719p^8 + 164\,358p^9.$$

Appendix B. Low-temperature polynomials

Free energy equation (5)

$$\psi_2(y) = y$$
$$\psi_4(y) = 2 - (1/2)y^2$$
$$\psi_5(y) = 4y$$
$$\psi_6(y) = 4 - 5y + 6y^2 + (1/3)y^3$$

$\psi_7(y) = 12y - 16y^2 + 4y^3$
$\psi_8(y) = 9 - 6y + 25y^2 - 24y^3 + (3/4)y^4$
$\psi_9(y) = 16y - 4y^2 + 52y^3 - 16y^4$
$\psi_{10}(y) = 24 + 17y - 46y^2 - 3y^3 + 72y^4 - (19/5)y^5$
$\psi_{11}(y) = 16y + 212y^2 - 264y^3 - 72y^4 + 48y^5$
$\psi_{12}(y) = (224/3) + 1000y - (763/2)y^2 + 948y^3 - 464y^5 - 154y^5 + (71/6)y^6$
$\psi_{13}(y) = 76y + 836y^2 - 1928y^3 + 2392y^4 - 252y^5 - 116y^6$
$\psi_{14}(y) = 260 + 323y - 752y^2 + 3004y^3 - 5732y^4 + 3427y^5 + 160y^6 - (209/7)y^7$
$\psi_{15}(y) = 580y + 1468y^2 - (5480/3)y^3 + 7616y^4 - 11828y^5 + 2468y^6 + 224y^7$
$\psi_{16}(y) = (1961/2) + 942y + 1212y^2 - 2886y^3 + (4809/2)y^4 + 18\,492y^5 - 15\,559y^6 + 516y^7 + (523/8)y^8$
$\psi_{17}(y) = 3432y + 376y^2 + 21\,256y^3 - 42\,268y^4 + 13\,860y^5 + 37\,908y^6 - 11\,584y^7 - 288y^8$
$\psi_{18}(y) = (11\,752/3) + 3017y + 14\,748y^2 - (139\,802/3)y^3 + 144\,610y^4 - 168\,599y^5 + 6404y^6 + 50\,961y^7 - 3936y^8 - (1115/8)y^9$
$\psi_{19}(y) = 17816 - 3608y + 115\,104y^2 - 312\,228y^4 + 560\,704y^5 - 374\,836y^6 - 49\,432y^7 + 39\,392y^8 - 64y^9.$

Magnetization, sublattice A, equation (6)

$\mu_4(y) = 2$
$\mu_5(y) = 8y$
$\mu_6(y) = 8 - 8y + 12y^2$
$\mu_7(y) = 24y - 32y^2 + 8y^3$
$\mu_8(y) = 34 + 8y + 52y^2 - 48y^3 + 2y^4$
$\mu_9(y) = 72y + 80y^2 + 120y^3 - 32y^4$
$\mu_{10}(y) = 152 + 104y - 220y^2 + 128y^3 + 168y^4 - 8y^5$
$\mu_{11}(y) = 344y + 976y^2 - 1216y^3 - 96y^4 + 112y^5$
$\mu_{12}(y) = 714 + 584y - 868y^2 + 4256y^3 - 2034y^4 - 376y^5 + 28y^6$
$\mu_{13}(y) = 2160y + 3808y^2 - 6784y^3 + 10\,496y^4 - 1176y^5 - 304y^6$
$\mu_{14}(y) = 3472 + 2704y + 1352y^2 + 10\,496y^3 - 23\,264y^4 + 14\,344y^5 + 388y^6 - 80y^7$
$\mu_{15}(y) = 13\,584y + 11\,488y^2 + 7344y^3 + 23\,424y^4 - 50\,264y^5 + 9712y^6 + 696y^7$
$\mu_{16}(y) = 17\,318 + 12\,912y + 31\,816y^2 - 19\,744y^3 + 65\,472y^4 + 67\,688y^5 - 64\,648y^6 + 1728y^7 + 208y^8$
$\mu_{17}(y) = 82\,352y + 46\,144y^2 + 168\,528y^3 - 303\,040y^4 + 183\,368y^5 + 157\,056y^6 - 44\,712y^7 - 1248y^8$
$\mu_{18}(y) = 88\,048 + 65\,264y + 240\,152y^2 - 147\,536y^3 + 947\,296y^4 - 1195\,792y^5 + 178\,708y^6 + 211\,528y^7 - 12\,304y^8 - 496y^9$
$\mu_{19}(y) = 487\,376y + 279\,808y^2 + 887\,744y^3 - 1546\,624y^4 + 3639\,400y^5 - 2585\,280y^6 - 111\,392y^7 + 150\,400y^8 + 1632y^9.$

Magnetization, sublattice B, equation (7)

$$\mu_2(y) = y$$
$$\mu_4(y) = 2 - y^2$$
$$\mu_5(y) = 4y$$
$$\mu_6(y) = 8 - 7y + 12y^2 + y^3$$
$$\mu_7(y) = 36y - 32y^2 + 12y^3$$
$$\mu_8(y) = 34 - 30y + 76y^2 - 72y^3 + 3y^4$$
$$\mu_9(y) = 120y - 72y^2 + 164y^3 - 64y^4$$
$$\mu_{10}(y) = 152 + 9y + 96y^2 - 59y^3 + 288y^4 - 19y^5$$
$$\mu_{11}(y) = 440y + 504y^2 - 544y^3 - 288y^4 + 240y^5$$
$$\mu_{12}(y) = 714 + 352y - 753y^2 + 2968y^3 - 1704y^4 - 762y^5 + 71y^6$$
$$\mu_{13}(y) = 2236y + 4200y^2 - 7520y^3 + 9504y^4 - 1244y^5 - 696y^6$$
$$\mu_{14}(y) = 3472 + 2137y - 1860y^2 + 18\,976y^3 - 25\,832y^4 + 16\,999y^5 + 968y^6 - 209y^7$$
$$\mu_{15}(y) = 13\,372y + 16\,648y^2 - 21\,544y^3 + 51\,352y^4 - 60\,812y^5 + 14\,656y^6 + 1584y^7$$
$$\begin{aligned}\mu_{16}(y) = {} & 17\,318 + 10\,642y + 16\,120y^2 + 34\,998y^3 - 42\,520y^4 + 116\,432y^5 - 92\,942y^6 \\ & + 3380y^7 + 525y^8\end{aligned}$$
$$\begin{aligned}\mu_{17}(y) = {} & 81\,008y + 57\,840y^2 + 68\,800y^3 - 75\,160y^4 - 25\,756y^5 + 240\,648y^6 - 79\,256y^7 \\ & - 2440y^8\end{aligned}$$
$$\begin{aligned}\mu_{18}(y) = {} & 88\,048 + 53\,589y^2 + 196\,720y^3 - 69\,610y^3 + 646\,812y^4 - 656\,865y^5 - 64\,256y^6 \\ & + 354\,383y^7 - 29\,588y^8 - 1133y^9\end{aligned}$$
$$\begin{aligned}\mu_{19}(y) = {} & 482\,160y + 255\,376y^2 + 830\,536y^3 - 1654\,152y^4 + 3036\,568y^5 - 1933\,000y^6 \\ & - 399\,968y^7 + 303\,384y^8 + 264y^9.\end{aligned}$$

Appendix C. Low-temperature series for $D = 0$

n	Λ	$2M_A$	M_B
0	1	1	1
1	0	0	0
2	1	0	−1
3	0	0	0
4	2	−2	−1
5	4	−8	−4
6	7	−12	−14
7	4	0	−16
8	12	−48	−11
9	56	−240	−148
10	85	−324	−467
11	16	−120	−352
12	237	−2 304	−886
13	1 092	−8 200	−6 480
14	1 182	−9 412	−14 651

15	244	−15 984	−15 256
16	7 609	−112 750	−63 953
17	24 492	−288 448	−265 684
18	19 101	−374 868	−518 100
19	27 276	−1 203 064	−921 168
20	258 001	−5 143 744	−3 596 879
21	588 340	−11 007 624	−10 988 824
22	469 190	−19 887 956	−22 250 017
23	1 721 868	−71 512 496	−54 492 168
24	8 574 255	−229 434 686	−182 361 304
25	15 587 260	−480 799 472	−485 872 424
26	19 019 487	−1 140 201 068	−1 099 778 395
27	83 601 404	−3 818 148 664	−3 023 514 544
28	286 002 711	−10 527 536 172	−9 047 912 433
29	483 837 524	−23 681 519 160	−23 220 720 020
30	925 796 394	−64 186 652 348	−57 661 206 008
31	3 632 190 036	−196 074 568 848	−162 009 811 552
32	9 935 584 271	−509 205 503 488	−455 566 252 239
33	17 937 247 396	−1 246 489 237 712	−1 175 640 422 620
34	45 021 389 115	−3 516 154 661 676	−3 079 814 836 505
35	151 664 076 748	−10 051 390 214 824	−8 597 948 883 880
36	369 822 320 783	−25 898 036 403 930	−23 492 159 524 108
37	758 927 807 732	−67 327 724 690 144	−61 635 502 786 700
38	2 124 327 429 138	−189 819 956 999 488	−165 691 566 038 227
39	6 332 313 107 532	−522 844 994 692 928	−457 792 286 802 656
40	14 894 179 490 640	−1 365 016 493 223 132	−1 237 997 546 964 335
41	34 463 262 202 180	−3 666 750 501 726 048	−3 298 093 488 627 480
42	98 319 436 258 400	−10 219 254 050 898 392	−8 963 468 060 434 086
43	270 356 528 568 500	−27 702 002 405 327 056	−24 573 787 094 876 248
44	643 330 125 661 066	−73 539 477 884 922 270	−66 362 713 706 273 013

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